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SOME EXTREMAL PROPERTIES OF GENERALIZED TCHEBYSHEV POLYNOMIALS

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In this note we prove several extremal properties of the Generalized Tchebychev Polynomials $\{T_{n,K}(z)\}$ respect to the minimax serie $S^K(f) = \sum_{s=0}^{\infty} E_s(f)_K$, where $K \subset \mathbb{C}$ is compact and $E_s(f)_K = \min_{P \in \Pi_n} \|f - P\|_{\infty,K}$.

1. Introduction.

Let $K \subseteq \mathbb{C}$ be a compact set containing infinitely many points and suppose that $\overline{\mathbb{C}} - K$ is connected. Then Mergelyan's theorem implies that $\lim_{s\to\infty} E_s(f)_K = 0$ for all f holomorphic in the interior of K and continuous in K. We consider the space of functions for which the minimax serie $S^K(f) = \sum_{s=0}^{\infty} E_s(f)_K$ is finite. We must be careful with interpretations of this serie as a measure of the kindness of f to be approximated by polynomials. For example, setting $K = [0, 2]$, we have $S^{[0,2]}(ax) = |a|$ and $S^{[0,2]}(x^n) \geq 2^{n-1}$ although it is clear that polynomials are the best functions to be approximated by polynomials. In this note, we are interested in bounds of $S^K(P_n)$ for certain sequences of polynomials $\{P_n = a_{0n}z^n + \ldots + a_{00}\}$. Our bounds will depend on n and a_{0n}. In particular, the generalized Tchebychev polynomials will be extremal respect to the minimax serie S^K as they are respect to the Tchebychev's norm $\| \|_{\infty,K}$.

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The minimax serie $S(f) = \sum_{n=0}^{\infty} E_n(f)_{[-1,1]}$ appears in the theory of equivalent norms on the Besov spaces $B^{\alpha}_{p,q}[-1,1]$ for the case $p = \infty, q = 1 = \alpha$ (cf. [3]). It is fairly possible to consider several classes of function spaces defined via variations of minimax series, such as $S^K_{p,q,(b_n)}(f) = \{\sum_{n=0}^{\infty} b_n E_{n,p}(f)^q_K\}^{\frac{1}{q}}$ (where $E_{n,p}(f)_K$ denotes the error of approximation to f by algebraic polynomials in L_p norm). We call all these spaces "Generalized Besov spaces" (see [1] for a reference).

2. Main results.

The generalized Tchebychev polynomials over K are defined as the monic polynomials $\widetilde{T}_{n,K}$ which satisfy the formula $\|\widetilde{T}_{n,K}\|_{\infty,K} = E_{n-1}(z^n)_K$. We will also use the formula $T_{n,K} = \dfrac{\widetilde{T}_{n,K}}{\|\widetilde{T}_{n,K}\|}$.

Theorem 2.1. *1) $S^K(\widetilde{T}_{n,K}) = n\|\widetilde{T}_{n,K}\|_{\infty,K} \leq S^K(\widetilde{P})$ for all $\widetilde{P}$ monic of degree n.*

2) $S^K(T_{n,K}) = n \geq S^k(P)$ for all $P \in \Pi_n$ such that $\|P\|_{\infty,K} = 1$.

Proof. Let $s \leq n-1$. Then

$$\|\widetilde{T}_{n,K}\|_{\infty,K} \geq E_s(\widetilde{T}_{n,K})_K \geq E_{n-1}(\widetilde{T}_{n,K})_K = E_{n-1}(z^n)_K = \|\widetilde{T}_{n,K}\|_{\infty,K}$$

so that $S^K(\widetilde{T}_{n,K}) = n\|\widetilde{T}_{n,K}\|$. Let $\widetilde{P}$ be monic of degree n.
Then $nE_{n-1}(\widetilde{P})_K \leq S^K(\widetilde{P})$. But $E_{n-1}(\widetilde{P})_K = E_{n-1}(z^n)_K = \|\widetilde{T}_{n,K}\|_{\infty,K}$ and the first assertion follows.

On the other hand, $S^K(T_{n,K}) = \dfrac{S^K(\widetilde{T}_{n,K})}{\|\widetilde{T}_{n,K}\|_{\infty,K}} = n$ and, if $P \in \Pi_n$ satisfy $\|P\|_{\infty,K} = 1$, then $S^K(P) \leq nE_0(P)_K \leq n\|P\|_{\infty,K} = n$. □

Let $G \subset \mathbb{C}$ be a bounded domain and set $K = \overline{G}$. Let $w : G \to \mathbb{R}^+$ be a weight function and set $L^2_w(G) = \left\{f \in H(G) : \iint_G |f(z)|^2 w(z)\,dz < \infty\right\}$. We denote by $\overline{P}_n(w) = \gamma_n(w)z^n + \cdots$ the n-th orthonormal polynomial respect to w with $\gamma_n(w) > 0$, by $P_n(w) = \alpha_n(w)z^n + \ldots$ the n-th orthogonal polynomial respect to w normalized by $\|P_n(w)\|_{\infty,K} = 1$, $\alpha_n(w) > 0$ and by $\mu_0(w) = \iint_G w(z)\,dz$ the 0-moment respect to w. For several choices of K the Tchebychev polynomials $T_{n,K}$ are orthogonal which respect to some measure on K or in its boundary ∂K so that in this context the above theorem can be seen as a result on extremality of Tchebychev polynomials between the systems of orthogonal monic polynomials (orthogonal polynomials normalized by $\|P\|_{\infty,K} = 1$, respectively). But then it is natural to ask if orthonormal Tchebychev polynomials are also extremal respect to the minimax serie.

Theorem 2.2. *With the notation above introduced, let $w_1, w_2 : G \to \mathbb{R}^+$ be two weight functions. Then the following bounds holds:*

1) $n\dfrac{\alpha_n(w_1)}{\gamma_n(w_2)}\mu_0^{-\frac{1}{2}}(w_2) \le S^K\big(P_n(w_1)\big)$,

2) $n\dfrac{\gamma_n(w_1)}{\gamma_n(w_2)}\mu_0^{-\frac{1}{2}}(w_2) \le S^K\big(\overline{P}_n(w_1)\big) \le n\dfrac{\gamma_n(w_1)}{\alpha_n(w_1)}$.

Proof. Let $w_2 : G \to \mathbb{R}^+$ be a weight function and denote by $E_s^{(2)}(w, f)$ the best approximation error of f by polynomials of degree at most s in $L_w^2(G)$ (i.e. $E_s^{(2)}(w, f) = \min_{p\in\Pi_s}\left\{\iint_G |f(z) - p(z)|^2 w(z)\,dz\right\}^{\frac{1}{2}}$). Then

$$E_s^{(2)}(w, f) \le E_s(f)_K\,\mu_0^{\frac{1}{2}}(w)$$

so that, setting $w = w_2$ and $f = P_n(w_1)$, we have

$$n\mu_0^{-\frac{1}{2}}(w_2)E_{n-1}^{(2)}\big(w_2, P_n(w_1)\big) \le nE_{n-1}\big(P_n(w_1)\big)_K \le S^K\big(P_n(w_1)\big).$$

An algebraic manipulation proves that

$$E_{n-1}^{(2)}\big(w_2, P_n(w_1)\big) = \alpha_n(w_1)E_{n-1}^{(2)}(w_2, z^n) = \frac{\alpha_n(w_1)}{\gamma_n(w_2)}$$

and the first assertion follows. The same argument proves

$$n\frac{\gamma_n(w_1)}{\gamma_n(w_2)}\mu_0^{-\frac{1}{2}}(w_2) \le S^K\big(\overline{P}_n(w_1)\big).$$

On the other hand, using Theorem 2.1 it is clear that

$$S^K\big(\overline{P}_n(w_1)\big) = \frac{\gamma_n(w_1)}{\alpha_n(w_1)}S^K\big(P_n(w_1)\big) \le n\frac{\gamma_n(w_1)}{\alpha_n(w_1)}\,. \qquad \square$$

Corollary 2.3. *Set $K = [-1, 1]$ and let w be a weight function on $[-1, 1]$. Then*

$$S^{[-1,1]}\big(\overline{P}_n(w)\big) \ge n\mu_0^{-\frac{1}{2}}(w) = O(n).$$

Furthermore,

$$S^{[-1,1]}(\overline{T}_n) = \frac{2}{\pi}n = O(n).$$

Proof. Take $w_1 = w_2 = w$ in Theorem 2.2. For the second claim it is enough to observe that $\overline{T}_n = \frac{2}{\pi} T_n$. $\square$

Corollary 2.4.

$$\frac{\alpha_n(w_1)}{\gamma_n(w_2)} \mu_0^{-\frac{1}{2}}(w_2) \leq 1.$$

Proof. By Theorems 2.1, 2.2 it is clear that

$$n \frac{\alpha_n(w_1)}{\gamma_n(w_2)} \mu_0^{-\frac{1}{2}}(w_2) \leq S^K\big(P_n(w_1)\big) \leq n. \qquad \square$$

Corollary 2.5.

$$S^k\big(\overline{P}_n(w)\big) \leq n \frac{1}{\sqrt{\min\limits_{\substack{\|P\|_{\infty,K}=1 \\ \partial P = n}} \iint_G |P(z)|^2 w(z)\, dz}} \,.$$

Proof. Orthonormality of $\{\overline{P}_n(w_1)\}_{n=0}^{\infty}$ implies

$$1 = \iint_G |\overline{P}_n(w_1)(z)|^2 w_1(z)\, dz = \left[\frac{\gamma_n(w_1)}{\alpha_n(w_1)}\right]^2 \iint_G |P_n(w_1)(z)|^2 w_1(z)\, dz.$$

Hence

$$\left[\frac{\alpha_n(w_1)}{\gamma_n(w_1)}\right]^2 \geq \left[\min_{\substack{\|P\|_{\infty,K}=1 \\ \partial P = n}} \iint_G |P(z)|^2 w(z)\, dz\right]$$

and the proof follows. $\square$

In the case $K = [-1, 1]$, the Christoffel functions

$$\lambda_n(w, x) = \left[\sum_{k=0}^{n-1} |\overline{P}(w, x)|^2\right]^{-1}$$

satisfy the formula

$$\lambda_n(w, x) = \min_{P(x)=1, \partial P = n} \int_{[-1,1]} |P(z)|^2 w(z)\, dz. \tag{2.1}$$

We may use this formula to prove the following corollary

Corollary 2.6. *Set* $K = [-1, 1]$. *Then*

$$S^{[-1,1]}\big(\overline{P}_n(w)\big) \le n\|\lambda_{n+1}(w, x)\|_\infty^{\frac{1}{2}}.$$

Proof. The Christoffel functions satisfy the formula

$$\lambda_n(w, x) = \min_{P(x)=1, \partial P=n} \int_{[-1,1]} |P(z)|^2 w(z)\, dz \le$$

$$\le \min_{\substack{\|P\|_{\infty,[-1,1]}=1 \\ \partial P=n}} \int_{[-1,1]} |P(z)|^2 w(z)\, dz.$$

The corollary follows applying the Corollary 2.5. $\square$

The class of weight functions $w(x)$ for which $[n\lambda_{n+1}(w, x)]^{-1} = O(1)$ uniformly in $[-1, 1]$ is of particular interest in Approximation Theory because of a theorem of G. Freud which states that for all these weights the Fourier expansions in orthogonal polynomials converge uniformly in $[-1, 1]$ to all continuous functions, and it has been studied by many authors (see [8] for a survey on this and other related subjects). For this class we obtain the following corollary

Corollary 2.7. *Set* $K = [-1, 1]$ *and suppose that* $[n\lambda_{n+1}(w, x)]^{-1} = O(1)$ *uniformly in* $[-1, 1]$. *Then* $O(n) \le S^{[-1,1]}\big(\overline{P}_n(w)\big) \le O(n^{\frac{3}{2}})$.

Proof. It follows from Corollary 2.6. $\square$

3. Final remarks.

One other possibility to define "generalized Besov spaces" is as follows (see [7] for a reference). Let $(X, \|\ \|)$ be a Banach space, let $\{X_n\}_{n=0}^\infty$ be a sequence of subspaces of X satisfying $0 = X_0 \subset X_1 \subset \dots$ and let $\beta = \{b_n\}_{n=0}^\infty$ be a sequence of positive numbers. Let $f \in X$ and denote by $E_n^X(f)$ the best approximation error of f approximating with elements of X_n. Then the corresponding generalized Besov spaces are defined by

$$X_q^\beta = \{f \in X : \|f\|_{X_q^\beta} = \|\{b_n E_n^X(f)\}_{n=0}^\infty\|_q < \infty\}, \quad 1 \le q \le \infty.$$

In what follows we assume that $\Pi \subset X$ and $X_n = \Pi_n$ for all n. Generalized Tchebychev polynomials also admits another definition. We say that $\widetilde{T}_n^X(z) \in \Pi_n$ is a generalized Tchebychev polynomial of degree n if $\partial \widetilde{T}_n^X(z) = n$, $\widetilde{T}_n^X(z)$ is monic and $\|\widetilde{T}_n^X(z)\| = E_{n-1}^X(z^n)$. With all these notations, Theorem 2.1 can be generalized as follows.

Theorem 3.1.

$$\widetilde{T}_n^{X_q^\beta}(z) = \widetilde{T}_n^X(z).$$

Proof. Let $s \le n-1$. Then

$$\|\widetilde{T}_n^X(z)\| \ge E_s^X\big(\widetilde{T}_n^X(z)\big) \ge E_{n-1}^X\big(\widetilde{T}_n^X(z)\big) = E_{n-1}^X(z^n) = \|\widetilde{T}_n^X(z)\|.$$

Hence $\|\widetilde{T}_n^X(z)\| = E_s^X\left(\widetilde{T}_n^X(z)\right)$ for all $s \le n-1$ and

$$\|\widetilde{T}_n^X(z)\|_{X_q^\beta} = \left[\sum_{s=0}^{n-1} b_s^q\right]^{\frac{1}{q}} \|\widetilde{T}_n^X(z)\|.$$

Let $\widetilde{P}(z)$ be monic of degree n. Then $E_s^X\big(\widetilde{P}(z)\big) \ge E_{n-1}^X\big(\widetilde{P}(z)\big) = E_{n-1}^X(z^n) = \|\widetilde{T}_n^X(z)\|$ for all $s \le n-1$. Hence

$$\|\widetilde{P}(z)\|_{X_q^\beta} = \left\{\sum_{s=0}^{n-1} b_s^q E_s^X\left(\widetilde{P}(z)\right)^q\right\}^{\frac{1}{q}} \ge \left[\sum_{s=0}^{n-1} b_s^q\right]^{\frac{1}{q}} \|\widetilde{T}_n^X(z)\| = \|\widetilde{T}_n^X(z)\|_{X_q^\beta}.$$

□

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